

Presentation of finite subgroups of mapping class group of genus 2 surface by Dehn twists

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Abstract

In this note we give presentations of all finite subgroups of the mapping class group of a closed surface of genus 2 by the Humphries generators up to conjugacy.

1 Introduction

Let \mathcal{MCG}_g denote the mapping class group of a closed orientable surface of genus g . The Dehn-Lickorish theorem states that \mathcal{MCG}_g is generated by Dehn twists along finitely many simple closed curves. S. P. Humphries found $2g + 1$ Dehn twists which generate \mathcal{MCG}_g for $g > 1$. A finite presentation in the Humphries generators was given by Wajnryb [7].

The objective of this note is to give a generator system represented by products of Humphries generators $\{\omega_j\}_{j=1}^5$ for all finite subgroups, up to topological conjugacy, of the mapping class group \mathcal{MCG}_2 of genus 2. In general, it is not easy to find finite subgroups of a group when only one of its presentations is given. For \mathcal{MCG}_g , due to the Nielsen realization theorem (S. Kerckhoff [5]), its finite subgroup is represented by a group of holomorphic automorphisms on a closed Riemann surface of genus g . S. A. Broughton [2] made a complete list of all groups, up to topological conjugacy, which arise as groups of holomorphic automorphisms on some closed Riemann surfaces of genus 2. It is natural to ask how those groups are embedded in \mathcal{MCG}_2 .

Our main theorem represents generators by products of ω_i ($i = 1, \dots, 5$) for each group in Broughton's list. For its statement we introduce some notation. Let a finite group G act on a Riemann surface R as a group of holomorphic automorphisms. If the genus of the factor surface R/G is h and the covering map $\pi : R \rightarrow R/G$ is branched over n points p_1, \dots, p_n with branching orders m_j , then $(h; m_1, \dots, m_n)$ is the *type* of the orbifold R/G . In stead of $(h; m_1, \dots, m_n)$, we often write $(h; \nu_1^{r_1}, \dots, \nu_p^{r_p})$ if ν_j appears r_j times in (m_1, \dots, m_n) . Let $\zeta_0, \zeta_1, \dots, \zeta_4$ be as in the following table. They have the orders indicated in the table.

order	
2	$\zeta_0 = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$
6	$\zeta_1 = \omega_1\omega_2\omega_3\omega_4\omega_5$
6	$\zeta_2 = \omega_1\omega_2\omega_4^{-1}\omega_5^{-1}$
8	$\zeta_3 = \omega_1^2\omega_2\omega_3\omega_4$
10	$\zeta_4 = \omega_1\omega_2\omega_3\omega_4$

The list below shows the group G_* corresponding to (2.*) in [2], the order $|G_*|$ and the orbifold type.

Theorem 1.1 *A non-trivial finite subgroup of \mathcal{MCG}_2 of a closed orientable surface of genus 2 is conjugate with one of the groups in the following list.*

- (2.a) $G_a = \langle x = \zeta_0 : x^2 = 1 \rangle \cong \mathbb{Z}_2, 2, (0; 2^6).$
- (2.b) $G_b = \langle x = \zeta_1^3 : x^2 = 1 \rangle \cong \mathbb{Z}_2, 2, (1; 2^2).$
- (2.c) $G_c = \langle x = \zeta_1^2 : x^3 = 1 \rangle \cong \mathbb{Z}_3, 3, (0; 3^4).$
- (2.e) $G_e = \langle x = \zeta_3^2 : x^4 = 1 \rangle \cong \mathbb{Z}_4, 4, (0; 2^2, 4^2).$
- (2.f) $G_f = \langle x = \zeta_0, y = \zeta_1^3 : x^2 = y^2 = [x, y] = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2, 4, (0; 2^5).$
- (2.h) $G_h = \langle x = \zeta_4^2 : x^5 = 1 \rangle \cong \mathbb{Z}_5, 5, (0; 5^3).$
- (2.i) $G_i = \langle x = \zeta_1 : x^6 = 1 \rangle \cong \mathbb{Z}_6, 6, (0; 3, 6^2).$
- (2.k1) $G_{k1} = \langle x = \zeta_2 : x^6 = 1 \rangle \cong \mathbb{Z}_6, 6, (0; 2^2, 3^2).$
- (2.k2) $G_{k2} = \langle x = \zeta_1^3, y = (\omega_4\zeta_2\omega_4^{-1})^2 : x^2 = y^3 = 1, xyx^{-1} = y^{-1} \rangle \cong D_3, 6, (0; 2^2, 3^2).$
- (2.l) $G_l = \langle x = \zeta_3 : x^8 = 1 \rangle \cong \mathbb{Z}_8, 8, (0; 2, 8, 8).$
- (2.m) $G_m = \langle x = \zeta_2^{-1}\zeta_3^2\zeta_2, y = \zeta_3^2 : x^4 = y^4 = 1, x^2 = y^2, xyx^{-1} = y^{-1} \rangle \cong \tilde{D}_2, 8, (0; 4, 4, 4).$
- (2.n) $G_n = \langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1} \rangle \cong D_4, \text{ where } x = \zeta_1^3, y = (\omega_4\omega_2^{-1}\omega_1^{-1})\zeta_3^2(\omega_4\omega_2^{-1}\omega_1^{-1})^{-1}, 8, (0; 2^3, 4).$
- (2.o) $G_o = \langle x = \zeta_4 : x^{10} = 1 \rangle \cong \mathbb{Z}_{10}, 10, (0; 2, 5, 10).$
- (2.p) $G_p = \langle x = \zeta_0, y = \zeta_1 : x^2 = y^6 = [x, y] = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6, 12, (0; 2; 6, 6).$
- (2.r) $G_r = \langle x, y : x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle \cong D_{4,3,-1},$
where $x = (\omega_3\omega_5^{-1}\omega_1^{-1})\zeta_3^6(\omega_3\omega_5^{-1}\omega_1^{-1})^{-1}, y = \zeta_1^4, 12, (0; 3, 4^2).$
- (2.s) $G_s = \langle x = \zeta_1^3, y = \omega_4\zeta_2\omega_4^{-1} : x^2 = y^6 = 1, xyx^{-1} = y^{-1} \rangle \cong D_6, 12, (0; 2^3, 3).$
- (2.u) $G_u = \langle x, y : x^2 = y^8 = 1, xyx^{-1} = y^3 \rangle \cong D_{2,8,3}, \text{ where } x = \zeta_1^3, y = (\omega_4\omega_2^{-1}\omega_1^{-1})\zeta_3(\omega_4\omega_2^{-1}\omega_1^{-1})^{-1}, 16, (0; 2, 4, 8).$

$$\begin{aligned}
(2.w) \quad G_w &= \left\langle x, y, z, w : \begin{array}{l} x^2 = y^2 = z^2 = w^3 = [y, z] = [y, w] = [z, w] = 1 \\ xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1} \end{array} \right\rangle, \\
&\text{where } x = (\omega_1\omega_2\omega_1)\zeta_1^3(\omega_1\omega_2\omega_1)^{-1}, y = \zeta_0, z = \zeta_1^3, w = \zeta_1^4, \\
&G_w \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3), 24, (0; 2, 4, 6). \\
(2.x) \quad G_x &= \langle x = \zeta_2^4, y = \zeta_3^2 : x^3 = y^4 = 1, xy^2 = y^2x, (xy)^3 = 1 \rangle \cong SL_2(3), 24, \\
&(0; 3^2, 4) \\
(2.aa) \quad G_{aa} &= \left\langle x, y, u : \begin{array}{l} x^3 = y^4 = (xy)^3 = 1, xy^2 = y^2x, u^2 = xyx^{-1}y^2 \\ uxu^{-1} = y^{-1}x^{-1}y, yuy^{-1} = x^{-1}yx \end{array} \right\rangle \\
&\text{where } x = \zeta_2^4, y = \zeta_3^2, u = (\omega_1\zeta_1\omega_4\omega_2\omega_1^{-1})\zeta_3(\omega_1\zeta_1\omega_4\omega_2\omega_1^{-1})^{-1}, \\
&G_{aa} \cong GL_2(3), 48, (0; 2, 3, 8)
\end{aligned}$$

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2 Facts about mapping class groups

2.1 The mapping class group and its generators

Our basic reference for mapping class groups is Farb-Margalit's book [3], in particular, Sections 3, 4 and 7. The mapping class group \mathcal{MCG}_g is generated by the Humphries generators or Dehn twists $\omega_0, \omega_1, \dots, \omega_{2g}$ along the loops depicted in Figure 1 (See [3, Theorem 4.14].) We consider an auxiliary Dehn twist ω_{2g+1} along the loop c_{2g+1} . Then the following relations hold:

$$\omega_i\omega_j = \omega_j\omega_i \quad \text{if } |i - j| \geq 2, 1 \leq i, j \leq 2g + 1 \quad (1)$$

$$\omega_i\omega_{i+1}\omega_i = \omega_{i+1}\omega_i\omega_{i+1} \quad (1 \leq i \leq 2g) \quad (2)$$

$$(\omega_1\omega_2 \cdots \omega_{2g+1})^{2g+2} = 1 \quad (\text{a chain relation}) \quad (3)$$

$$\text{If } \zeta_0 = \omega_1\omega_2 \cdots \omega_{2g}\omega_{2g+1}\omega_{2g+1}\omega_{2g} \cdots \omega_2\omega_1, \text{ then } \zeta_0^2 = 1 \quad (4)$$

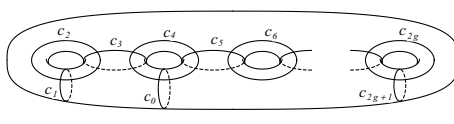


Figure 1

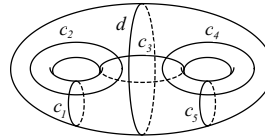


Figure 2

The element ζ_0 is the *hyperelliptic involution*. If $1 \leq k \leq m$, then

$$\begin{aligned}
\omega_{i+k}(\omega_i\omega_{i+1} \cdots \omega_{i+m}) &= (\omega_i\omega_{i+1} \cdots \omega_{i+m})\omega_{i+k-1}, \\
\omega_{i+k}^{-1}(\omega_i^{-1}\omega_{i+1}^{-1} \cdots \omega_{i+m}^{-1}) &= (\omega_i^{-1}\omega_{i+1}^{-1} \cdots \omega_{i+m}^{-1})\omega_{i+k-1}^{-1},
\end{aligned} \quad (5)$$

since we obtain from (1) and (2)

$$\begin{aligned}
\omega_{i+k}(\omega_i\omega_{i+1}\cdots\omega_{i+m}) &= \omega_i\omega_{i+1}\cdots(\omega_{i+k}\omega_{i+k-1}\omega_{i+k})\omega_{i+k+1}\cdots\omega_{i+m} \\
&= \omega_i\omega_{i+1}\cdots(\omega_{i+k-1}\omega_{i+k}\omega_{i+k-1})\omega_{i+k+1}\cdots\omega_{i+m} \\
&= (\omega_i\omega_{i+1}\cdots\omega_{i+m})\omega_{i+k-1}.
\end{aligned}$$

The second equation can be obtained in a similar way. Let

$$\zeta = \omega_1\omega_2\cdots\omega_{2g+1}, \quad \eta = \omega_1\omega_2\cdots\omega_{2g}.$$

Applying (5) we have $\omega_i\zeta = \zeta\omega_{i-1}$ for $i = 2, \dots, 2g+1$. Since

$$\omega_1\zeta = \zeta\zeta^{-1}\omega_1\zeta = \zeta\omega_{2g+1}^{-1}\omega_{2g}^{-1}\cdots\omega_3^{-1}\omega_2^{-1}\zeta = \zeta^2\omega_{2g}^{-1}\cdots\omega_2^{-1}\omega_1^{-1},$$

we have $\omega_1\zeta = \zeta^2\eta^{-1}$, $\omega_1\zeta^2 = \zeta^2\omega_{2g+1}$ and $\omega_1\zeta^3 = \zeta^2\omega_{2g+1}\zeta = \zeta^3\omega_{2g}$. Continuing in this way we have for $i, j = 1, \dots, 2g+1$,

$$\omega_i\zeta^j = \zeta^j\omega_{i-j} \quad (i \neq j), \quad \omega_i\zeta^i = \zeta^{i+1}\eta^{-1}, \quad (6)$$

where the index k for ω_k is considered modulo $2g+2$. From this follows

$$\omega_i = \zeta^{i+1}\eta^{-1}\zeta^{-i}, \quad (i = 1, 2, \dots, 2g+1). \quad (7)$$

Lemma 2.1 (special cases of [3, Proposition 4.12]) *If $\eta = \omega_1\omega_2\cdots\omega_{2g}$ and $\xi = \omega_1^2\omega_2\cdots\omega_{2g}$, then η^{2g+1} is a conjugate of ζ_0 in (4) and $\xi^{2g} = \eta^{2g+1}$. Hence $\eta^{4g+2} = \xi^{4g} = 1$.*

Proof. Note that $\zeta_0 = \zeta_0^{-1}$. From (7)

$$\begin{aligned}
\eta^{2g+1} &= (\zeta^{-1}\omega_1^{-1}\zeta^2)(\zeta^{-2}\omega_2^{-1}\zeta^3)\cdots(\zeta^{-2g-1}\omega_{2g+1}^{-1}) \\
&= \zeta^{-1}\omega_1^{-1}\omega_2^{-1}\cdots\omega_{2g+1}^{-1} \\
&= (\omega_{2g+1}\omega_{2g}\cdots\omega_2\omega_1\omega_1\omega_2\cdots\omega_{2g}\omega_{2g+1})^{-1}.
\end{aligned}$$

A consequence of (5) is $\omega_{i+1}\eta = \eta\omega_i$ ($i = 1, \dots, 2g-1$). So we have

$$\xi^{2g} = \underbrace{\omega_1\eta\omega_1\eta\cdots\omega_1\eta}_{2g \text{ times}} = \omega_1\omega_2\cdots\omega_{2g}\eta^{2g} = \eta^{2g+1}.$$

From (7) we see that \mathcal{MCG}_g is generated by three elements ω_0 , ζ and η . If $g = 2$, then $\omega_0 = \omega_5$. Therefore, we obtain a theorem by M. Korkmaz for $g = 2$.

Corollary 2.1 (Korkmaz [6]) *The mapping class group \mathcal{MCG}_2 is generated by ζ and η , where $\zeta^6 = \eta^{10} = 1$.*

2.2 Case of Genus 2

The mapping class group \mathcal{MCG}_2 is generated by Humphries generators $\omega_1, \omega_2, \omega_3, \omega_4$ and ω_5 with defining relations (1), (2) and

$$\begin{aligned}\zeta_1^6 &= (\omega_1\omega_2\omega_3\omega_4\omega_5)^6 = 1, \\ \zeta_0^2 &= (\omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1)^2 = 1, \\ \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1 &\text{ and } \omega_i \text{ commute for } i = 1, 2, 3, 4, 5.\end{aligned}\quad (8)$$

See [1, p.184]. Let ζ_0, \dots, ζ_3 and ζ_4 be as in the table of Section 1. From (8), the hyperelliptic involution $\zeta_0 = \omega_1\omega_2\omega_3\omega_4\omega_5^2\omega_4\omega_3\omega_2\omega_1$ equals any conjugate of itself or its inverse. We proved in Lemma 2.1 that $\zeta_3^4 = \zeta_4^5 = \zeta_0$, and hence $\zeta_3^8 = \zeta_4^{10} = 1$. We shall show $\zeta_2^6 = 1$ in Lemma 3.1, but topologically this arises from 2-chain relations [3, p.107] in one-holed tori on each side of the loop d in Figure 2. We remark that S. Hirose studied in [4] presentations of periodic mapping classes on orientable closed surfaces of genus ≤ 4 by Dehn twists by using topological and algebraic-geometric methods.

3 Proof of the main theorem

3.1 Abelian Groups

Our basic tools are elaborate applications of (6) for $g = 2$

$$\omega_i\zeta_1^j = \zeta_1^j\omega_{i-j} \quad (i \neq j), \quad \omega_i\zeta_1^i = \zeta_1^{i+1}\zeta_4^{-1}, \quad (9)$$

and trivial equations

$$\zeta_1\omega_5 = \zeta_0\omega_1^{-1}\omega_2^{-1}\omega_3^{-1}\omega_4^{-1}, \quad (10)$$

$$\omega_1^{-1}\omega_2^{-1}\omega_3^{-1}\omega_4^{-1}\omega_5^{-1} = \zeta_1\zeta_0^{-1} = \zeta_1\zeta_0. \quad (11)$$

Lemma 3.1 *If $\zeta_2 = \omega_1\omega_2\omega_4^{-1}\omega_5^{-1}$, then $\zeta_2^3 = \zeta_0$. Hence ζ_2 has order 6.*

Proof. The following equation deduced from (1), (9) and (11) implies $\zeta_2^3 = \zeta_0$.

$$\begin{aligned}\zeta_2 &= (\zeta_1\omega_5^{-1}\omega_4^{-1}\omega_3^{-1})(\omega_3\omega_2\omega_1\zeta_1\zeta_0) = \zeta_1\omega_2\omega_5^{-1}\omega_1\omega_4^{-1}\zeta_1\zeta_0 \\ &= \omega_3(\zeta_1\omega_5^{-1})(\omega_1\zeta_1)\omega_3^{-1}\zeta_0 = (\omega_3\zeta_4)\zeta_1^2(\omega_3\zeta_4)^{-1}\zeta_0.\end{aligned}\quad (12)$$

Proofs of Theorem 1.1 for cyclic groups $G_c = \langle \zeta_1^2 \rangle \cong \mathbb{Z}_3$, $G_e = \langle \zeta_3^2 \rangle \cong \mathbb{Z}_4$, $G_h = \langle \zeta_4^2 \rangle \cong \mathbb{Z}_5$, $G_l = \langle \zeta_3 \rangle \cong \mathbb{Z}_8$ and $G_o = \langle \zeta_4 \rangle \cong \mathbb{Z}_{10}$ are straightforward. Since ζ_0 is in the center of \mathcal{MCG}_2 , proofs for abelian groups $G_f = \langle \zeta_0, \zeta_1^3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_p = \langle \zeta_0, \zeta_1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ are also easy.

There are isomorphic pairs of groups (G_a, G_b) and (G_i, G_{k1}) . The types of orbifold for G_a and G_{k1} imply that they must contain the hyperelliptic involution. So $G_a = \langle \zeta_0 \rangle \cong \mathbb{Z}_2$, $G_b = \langle \zeta_1^3 \rangle \cong \mathbb{Z}_2$ and $G_i = \langle \zeta_1 \rangle \cong \mathbb{Z}_6$. The equation (12) means that $G_{k1} = \langle \zeta_2 \rangle \cong \langle \zeta_1^2 \rangle \times \langle \zeta_0 \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$.

3.2 Non-abelian Groups

Groups with two or more generators require much effort. Assume that x and y generate a finite group of \mathcal{MCG}_2 . We know each of them is a conjugate of a power of some ζ_i , ($i = 0, 1, \dots, 4$), but finding suitable conjugates of x and y so that their product has also a finite order is rather laborious.

3.2.1 Groups G_{k2} and G_s

Let $a = \zeta_1^3$ and $b = \omega_1\omega_2\omega_5^{-1}\omega_4^{-1} = \omega_4\zeta_2\omega_4^{-1}$. Then $a^2 = b^6 = 1$. It holds $(ba)^2 = baba^{-1} = 1$ because (9) yields

$$ba = \omega_1\omega_2\omega_5^{-1}\omega_4^{-1}\zeta_1^3 = \omega_1\omega_2\zeta_1^3\omega_2^{-1}\omega_1^{-1}.$$

So a and b generate $G_s \cong D_6$, and a and b^2 generate $G_{k2} \cong D_3$.

3.2.2 Groups G_n , G_u

The group $G_u = \langle x, y : x^2 = y^8 = 1, xyx^{-1} = y^3 \rangle$ has the subgroup

$$G_n = \langle x, z = y^2 : x^2 = z^4 = 1, xzx^{-1} = z^{-1} \rangle.$$

If $x = \zeta_1^3$ and $y = \omega_1\omega_2\omega_4\omega_3\omega_2 = (\omega_4\omega_2^{-1}\omega_1^{-1})\zeta_3(\omega_4\omega_2^{-1}\omega_1^{-1})^{-1}$, then $x^2 = 1$, $y^4 = \zeta_0$ and $y^8 = 1$. By using (8), (9) and (10),

$$\begin{aligned} xyx^{-1} &= \zeta_1^3(\omega_1\omega_2\omega_4\omega_3\omega_2)\zeta_1^{-3} &= \omega_4\omega_5\zeta_1(\zeta_1\omega_5)(\zeta_1\omega_3\omega_2)\zeta_1^{-3} \\ & &= \omega_4\omega_5\zeta_1(\zeta_0\omega_1^{-1}\omega_2^{-1}\omega_3^{-1}\omega_4^{-1})(\omega_4\omega_3)\zeta_1^{-2} \\ & &= \omega_4\omega_5\omega_2^{-1}\omega_3^{-1}\zeta_1^{-1}\zeta_0 \\ & &= (\omega_1\omega_2\omega_4\omega_3\omega_2)^{-1}\zeta_0 = (\omega_1\omega_2\omega_4\omega_3\omega_2)^3 = y^3. \end{aligned}$$

3.2.3 Groups G_r , G_w

Let $G_w = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ have the presentation as in (2.w). Since $[z, w] = 1$, $u = zw$ generates $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and satisfies $u^3 = z$ and $u^4 = w$. Therefore G_w can be written as

$$\langle x, y, u : x^2 = y^2 = u^6 = [x, y] = [y, u] = 1, xux^{-1} = u^{-1}y \rangle.$$

This is an extension of the abelian group $G_p = \langle y, u : y^2 = u^6 = [y, u] = 1 \rangle$. By letting $t = xu^3$ and $w = u^4$, we find the subgroup

$$G_r = \langle t, w : t^4 = w^3 = 1, twt^{-1} = w^{-1} \rangle$$

of G_w . Let $\zeta_5 = \omega_1\omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\omega_4^{-1}$. By using (1), (2), (9) and (11) we have

$$\begin{aligned} \zeta_5 &= \zeta_1\omega_5^{-1}\omega_4^{-1}\omega_3^{-1}\omega_1\omega_5^{-1}\omega_3\omega_2\omega_1\zeta_1\zeta_0 = \zeta_1\zeta_5\zeta_1\zeta_0 \\ &= \zeta_1\omega_2\omega_1\omega_2\omega_4^{-1}\omega_5^{-1}\omega_4^{-1}\zeta_1\zeta_0 = \omega_3(\zeta_1\omega_4^{-1}\omega_1\omega_5^{-1}\omega_2\zeta_1)\omega_3^{-1}\zeta_0 \\ &= \omega_3\omega_5^{-1}(\zeta_1\omega_5^{-1}\omega_1\zeta_1)\omega_1\omega_3^{-1}\zeta_0 \\ &= (\omega_3\omega_5^{-1}\omega_1^{-1})\zeta_3^2(\omega_3\omega_5^{-1}\omega_1^{-1})^{-1}\zeta_0 = (\omega_3\omega_5^{-1}\omega_1^{-1})\zeta_3^6(\omega_3\omega_5^{-1}\omega_1^{-1})^{-1}. \end{aligned}$$

Thus we obtain

$$\zeta_5^2 = \zeta_0, \quad \zeta_5^{-1}\zeta_1\zeta_5 = \zeta_1^{-1}\zeta_0. \quad (13)$$

and also

$$(\zeta_5\zeta_1^k)^2 = \zeta_0 \text{ if } k \text{ is even, } (\zeta_5\zeta_1^k)^2 = 1 \text{ if } k \text{ is odd.} \quad (14)$$

Let $x = \zeta_5\zeta_1^3 = (\omega_1\omega_2\omega_1)\zeta_1^3(\omega_1\omega_2\omega_1)^{-1}$, $y = \zeta_0$ and $u = \zeta_1$. Then $y^2 = u^6 = [x, y] = [y, u] = 1$. The relations $x^2 = 1$ and $xux^{-1} = u^{-1}y$ are equivalent to $(\zeta_5\zeta_1^3)^2 = 1$ and $(\zeta_5\zeta_1^4)^2 = \zeta_0$, which follow from (14).

3.2.4 Groups G_m , G_x , G_{aa}

$SL_2(3) = \langle x, y : x^3 = y^4 = 1, xy^2 = y^2x, (xy)^3 = 1 \rangle$, where

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{the entries are in } \mathbb{Z}/(3\mathbb{Z})).$$

$GL_2(3)$ is obtained by adding to $SL_2(3)$ the matrix

$$u = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix},$$

which satisfies $u^8 = 1$, $uy^2 = y^2u$ and

$$u^2 = xyx^{-1}y^2, \quad xux^{-1} = y^{-1}x^{-1}y, \quad uyu^{-1} = x^{-1}yx. \quad (15)$$

By letting $v = x^{-1}yx$, we find the subgroup G_m of $SL_2(3)$ presented by

$$G_m = \langle v, y : v^4 = y^4 = 1, v^2 = y^2, v y v^{-1} = y^{-1} \rangle.$$

Now, we represent x , y and u by $\omega_1, \dots, \omega_5$. By using (1) and (9)

$$\begin{aligned} \zeta_2\zeta_3^{-2} &= (\omega_1\omega_2\omega_4^{-1}\omega_5^{-1})(\omega_4^{-1}\omega_3^{-1}\omega_2^{-1}\omega_1^{-2})(\omega_5\zeta_1^{-1}\omega_1^{-1}) \\ &= \omega_1\omega_2(\omega_4^{-1}\zeta_1^{-1}\omega_5)(\omega_1^{-1}\zeta_1^{-1})\omega_1^{-1} \\ &= (\omega_1\omega_2)\zeta_1^{-2}(\omega_1\omega_2)^{-1}. \end{aligned}$$

Hence $(\zeta_2\zeta_3^{-2})^3 = 1$. Let $x = \zeta_2\zeta_0 = \zeta_2^4$ and $y = \zeta_3^2$. Then we have $x^3 = y^4 = 1$, $y^2 = \zeta_0$, $(xy)^3 = (\zeta_2\zeta_3^{-2})^3 = 1$ and $xy^2 = y^2x$. Let

$$\begin{aligned} a &= \omega_1^{-1}\zeta_2\omega_1 = \omega_2\omega_4^{-1}\omega_5^{-1}\omega_1 = \omega_2\omega_1\omega_4^{-1}\omega_5^{-1}, \\ b &= \omega_1^{-1}\zeta_3\omega_1 = \omega_1\omega_2\omega_3\omega_4\omega_1 = \zeta_4\omega_1 = \omega_2\omega_1\omega_2\omega_3\omega_4, \\ c &= \omega_2\omega_3\omega_5\omega_4\omega_3 = (\zeta_1\omega_4\omega_2^{-1}\omega_1^{-1})\zeta_3(\zeta_1\omega_4\omega_2^{-1}\omega_1^{-1})^{-1} \end{aligned}$$

and $u = \omega_1c\omega_1^{-1}$. Then $u^8 = 1$. Since $(x, y, u) = (\omega_1a\zeta_0\omega_1^{-1}, \omega_1b^2\omega_1^{-1}, \omega_1c\omega_1^{-1})$, the relations $xux^{-1} = y^{-1}x^{-1}y$ and $b^2ca(b^2c)^{-1} = a^{-1}$ are equivalent. By using (9) we have $\omega_1\zeta_4 = \zeta_4\omega_4^{-1}\omega_3^{-1}\omega_2^{-1}\zeta_4 = \zeta_4^2\omega_3^{-1}\omega_2^{-1}\omega_1^{-1}$, and then

$$b^2c = \zeta_4(\omega_1\zeta_4\omega_1\omega_2\omega_3)\omega_5\omega_4\omega_3 = \zeta_4^3\omega_5\omega_4\omega_3 = \zeta_1^3.$$

On the other hand, from $\omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\zeta_1^3 = \omega_2\omega_1\zeta_1^3\omega_1^{-1}\omega_2^{-1}$ we have

$$(\omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\zeta_1^3)^2 = 1.$$

Since $\zeta_1^3 = \zeta_1^{-3}$ this means $b^2ca(b^2c)^{-1} = a^{-1}$. The relations $uyu^{-1} = x^{-1}yx$ and $(ac)b^2(ac)^{-1} = b^2$ are equivalent. The last relation easily follows from

$$ac = \omega_2\omega_1\omega_4^{-1}\omega_5^{-1}\omega_2\omega_3\omega_5\omega_4\omega_3 = \omega_2\omega_1\omega_4^{-1}\omega_2\omega_3\omega_4\omega_3 = \omega_2\omega_1\omega_4^{-1}\omega_2\omega_4\omega_3\omega_4 = b.$$

Finally we show the first relation $u^2 = xyx^{-1}y^2$ in (15), which is equivalent to $c^2 = ab^2a^{-1}b^4$. Since $b^4 = b^{-4} = \zeta_0$, $ab^2a^{-1}b^4 = ab^{-2}a^{-1} = (ab^{-1}a^{-1})^2$. We obtain $c^2 = (ab^{-1}a^{-1})^2$ from

$$\begin{aligned} ab^{-1}a^{-1}\zeta_0 &= \omega_2\omega_1\omega_4^{-1}\omega_5^{-1}(\omega_1^{-1}\zeta_4^{-1}\zeta_0)\omega_5\omega_4\omega_1^{-1}\omega_2^{-1} \\ &= \omega_2\underline{\omega_1}\omega_4^{-1}(\underline{\omega_1^{-1}}\omega_5\omega_4\underline{\omega_3\omega_2\omega_1})\omega_5\omega_4\underline{\omega_1^{-1}\omega_2^{-1}} \\ &= \omega_2\omega_4^{-1}\omega_5\omega_4\omega_3\omega_5\omega_4 = \omega_2\omega_4^{-1}\omega_4\omega_5\omega_4\omega_3\omega_4 \\ &= \omega_2\omega_5\omega_3\omega_4\omega_3 = c. \end{aligned}$$

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